INTRODUCTION TO MEAN-REVERTING PROCESS
Although the mean-reverting phenomenon appears to violate the definition of independent events, it simply reflects the fact that the probability density function $P(x)$ of any random variable $x$, by definition, is nonnegative over every interval and integrates to one over the interval $(-\infty, \infty)$. Thus, as $x$ moves away from the mean, the proportion of the distribution that lies closer to the mean than $x$ increases continuously (Weisstein, 2012)

Formally, this can be stated as;

$$\int_{-\infty}^{\mu_{i}} P(x)dx > \int_{-\infty}^{\mu_{j}} P(x)dx$$

for $i > j > 0$

Mean reverting models originally proposed for specifying interest rate dynamics by Vasicek (1977) and it was then so-called as Vasicek model. This model is also referred as arithmetic Ornstein-Uhlenbeck process and described by the following SDE:

$$dX_t = (\mu - \beta X_t)dt + \sigma dW_t$$

$$= \beta (L - X_t)dt + \sigma dW_t$$

Mean reversion in this section is modelled by having a drift term that is negative if the spot electricity prices are higher than the mean reversion level and positive if it is lower. This representation is a one-factor model and it reverts to the long-term mean $L = \frac{\mu}{\beta}$ with $\beta$ being the magnitude of the speed of mean-reversion whereas the second term in the above representation is the volatility of the process.
The explicit solution to the SDE represented in Equation (17) above between any two periods \( s \) and \( t \), with \( 0 \leq s < t \), can be derived from the solution to the general Ornstein-Uhlenbeck SDE representation as follows;

\[
X_t = \theta \left(1 - e^{-\beta(t-s)}\right) + x_s e^{-\beta(t-s)} + \sigma e^{-\beta t} \int_s^t e^{\beta u} \, dW_u
\]

and the discrete time version of this equation, on a time grid \( 0 = t_0, t_1, t_2, \ldots \) with time step \( \Delta t = t_i - t_{i-1} \) is given by;

\[
x(t_i) = c + bx(t_{i-1}) + \delta \varepsilon(t_i)
\]

where the coefficients are; \( c = \theta(1 - e^{-\alpha \Delta t}) \) and \( b = e^{-\alpha \Delta t} \) and \( \varepsilon(t) \) is a Gaussian white noise \((\varepsilon \sim N(0,1))\).

Finally, the volatility of the innovations can be derived via Ito isometry as;

\[
\delta = \sigma \sqrt{(1-e^{-2\alpha \Delta t})/2\alpha}
\]

where if one used the Euler scheme to discretise the equation this would lead to \( \delta = \sigma \sqrt{\Delta t} \), which is the same (first order in \( \Delta t \)) as the above.

**MODELLING PRICES IN NEM WITH THE MEAN-REVERTING PROCESS**

Higgs and Worthington (2010) modelled electricity prices in NEM by mean-reverting process. The model they used also included a deterministic as well as a stochastic component. The deterministic component of their model accounted for predictable regularities while the stochastic component was derived following De Jong (2006). Their model employed five years of daily price data for NSW, QLD, VIC and SA regions. Their main conclusion was the existence of strong mean reversion in electricity prices after a price spike than in a normal
period, which is in parallel with international experience, and price volatility that is more than fourteen times higher in spike periods than in normal periods.

This study will build on the Higgs and Worthington’s findings by modelling the electricity spot prices with mean reverting stochastic processes and generate forecast values with Monte Carlo simulations. It then compares the forecast accuracy measures of the model with other stochastic models constructed in this thesis.

PARAMETER ESTIMATION OF THE MEAN-REVERTING MODEL

The discrete form of the mean-reverting process (EQUATION 19) is used to calibrate the model developed in this section. This discrete form is the exact formulation of an AR(1) process. Having $0 < b < 1$ when $0 < \alpha$ implies that this AR(1) process is stationary and mean-reverting to a long-term mean given by $\theta$. This can also be confirmed by computing mean and variance of the process as the distribution of $x(t)$ is Gaussian, it is characterized by its first two moments. The conditional mean and variance of $x(t)$ given $x(s)$ can be derived from the Ornstein-Uhlenbeck SDE as follows;

$$E_s[x_t] = \theta + (x_s - \theta)e^{-\alpha(t-s)}$$

$$Var_s[x_t] = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha(t-s)})$$

As time increases, the mean tends to the long-term value $\theta$ and the variance remains bounded (unlike geometric Brownian motion), implying mean reversion. In other words, the long-term distribution of the Ornstein-Uhlenbeck is stationary and is Gaussian with $\theta$ as mean and $\sqrt{\sigma^2 / 2\alpha}$ as standard deviation.

Modelling electricity prices with pure mean-reverting model is relatively simple (as compared to more advanced models like jump-diffusion models) as there are only three
parameters that need to be estimated. These parameters are; the speed of mean reversion, long-run mean and the measure of the volatility process.

The literature has a number of techniques in estimating the speed of mean reversion (weighted or unweighted autoregression, quasi-maximum likelihood approach, generalised method of moments or methods based on Laplace transform) however; the ordinary least squares (OLS) method is the only method that directly estimates the mean-reverting parameter whereas all other methods are based on the joint estimation of all parameters. These models are considered as more difficult to implement even though they may provide more efficient results (Gourieroux and Monfort, 2010). Therefore, estimating the speed of mean reversion in this study is performed via OLS estimation for analytical simplicity. This is attained by performing a linear regression between the log prices and their first difference as follows;

\[
\frac{dx_t}{dt} = -\beta t + \beta L + \sigma \frac{\sigma}{dt} dW_t
\]

The speed of mean reversion and the long-run mean level are calculated from the coefficients of a linear fit scaled by the time interval parameter. In-sample data (01/06/2006 to 31/05/2010) of the study is used to estimate the parameters of the model. In this unmodified (pure) model as in GBM model, volatility term is set to a constant, despite the fact that empirical evidence suggests that electricity prices exhibit heteroskedasticity. This is also done for analytical simplicity. The speed at which electricity prices revert to their long run levels may depend on several factors such as the weather, magnitude and direction of the price shocks.

The following table demonstrates the parameter estimates of the mean-reverting model for each region of the NEM. The mean reversion parameters are significant and positive for all
regions of the NEM and ranges from 100 to 157, highlighting the evolution of mean-reversion in electricity price processes in NEM. The mean reversion rate is always greater than, or equal to, zero and higher numbers correspond to faster mean reversion as in the parameter table described below.

<table>
<thead>
<tr>
<th>Table 1 Parameter Estimates of Mean-Reverting Model</th>
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<tbody>
<tr>
<td></td>
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<tr>
<td>mean reversion rate, $\beta$</td>
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<tr>
<td>measure of volatility, $\sigma$</td>
</tr>
</tbody>
</table>

As is pointed earlier, the speed of mean reversion parameter represents the annualised rate at which the underlying short-term price returns to its expected long-run equilibrium value. Hence the inverse of the speed of mean reversion rate gives the actual time scale over which mean reversion occurs. For example, a mean-reversion rate of 122.411 corresponds to an electricity price process whose price reverts to its expected value over the course of three days as is the case for NSW region\(^1\). In fact, the regions of QLD and SA have the greatest mean reversion rates in NEM signifying the relatively shorter number of days for spiked prices to return to long-run mean levels as compared to other regions in NEM. It is also interesting to note that these regions have the highest annualised volatility measures, which points to the fact that these regions maybe the least mature electricity markets in NEM.

**SIMULATION OF THE MEAN-REVERTING PROCESS**

In order to estimate the future quantities of interest ($\theta$) with the mean-reverting model, the Monte-Carlo simulations is used. The need for Monte Carlo simulations to estimate a future quantity of interest is with mean-reverting model is due to the difficulties of finding an explicit solution to the SDE or the distribution of $X_t$ is unknown.

\(^1\) $365/122.411 = 2.98$
Suppose that the computation of $\theta := E[f(X_T)]$ where $X_t$ satisfies:

$$dX_t = \beta(L - X_t)dt + \sigma dW_t$$

The solution to above equation is given by the following representation;

$$X_T = X_t + \exp(\beta T)[X_0 - X_t] + \sigma \exp(\beta T) \int_0^T \exp(\beta s) dW_s$$

It is significant to note that unlike the case in solutions to GBM process, $X_T$ now depends on the entire path of the Brownian motion. This means that the computation of an unbiased estimate of $\theta$ by first simulating the entire path of the Brownian motions is not possible since it is only possible to simulate the latter at discrete intervals of time. However, the distribution of $X_T$ is known as Gaussian. In particular, this places the issue to the context where if $\theta$ can not be computed analytically, then the estimation of it can be derived by simulating $X_T$ directly.

In deriving the mean-reverting models -developed for each region of the NEM- of this section, similar to construction of GBM models, MATLAB’s Hull-White-Vasicek (HWV) constructor is used. This constructor creates and displays HWV objects, which derive from SDE with drift rate expressed in mean-reverting form classes. The state variables in this constructor are driven by Brownian motion sources of risk over consecutive observation periods, approximating continuous-time HWV stochastic processes with Gaussian diffusions (Matlab 2012). The mean-reverting models constructed by MATLAB constructor for each region of the NEM are then simulated via MATLAB’s simBySolution method (this method is explained in the preceding section on GBM modeling).
The following chart illustrates a sample path generated via mean-reverting model for NSW region. As is seen, mean-reverting model generates random prices deviating from the long-run mean however on average these prices revert back to the long-run mean levels. This finding is very different from the earlier findings based on pure GBM model where the existence of mean-reversion was non-existing.

![A simulated path with mean-reverting model for NSW log-prices (100 days ahead)](image)

The following chart illustrates the electricity price forecast generated by the mean-reverting model for all regions of NEM for the same time horizon. As is seen, simulated price series quickly revert to their long-run equilibrium as indicated by the upward convex move. The upward convex move at the beginning of the forecast horizon shows how long it takes prices to reach to their long-run mean levels. As expected, this upward convex is greatest for the regions of QLD and TAS where the mean-reversion parameters were estimated to be the highest. Once the predicted prices reach their long-run mean levels, they tend to present rather a constant price level (albeit with minor fluctuations). This is the result of taking the mean of 10,000 simulated paths to derive a point forecast value.
The empirical evidence of modelling with mean-reverting model in NEM strengthens the inaptness of modelling electricity prices with this stochastic diffusion process. This is due to the fact that simulations based on mean-reverting model over 90 days horizon seem not to represent the true price paths of the electricity prices as the simulated paths fail to capture the jumps prevalent in the price series.

This is attributable to the assumptions of the mean-reverting process where the conditional distribution of the logarithm of the electricity prices is Gaussian, which rather underestimates the large movements in the price series. Another shortcoming of the predicting electricity prices with mean-reverting process is that it assumes non-negativity of the prices. The literature in Australian electricity prices as well as international markets pointed out the fact that there are negative price occurrences in the markets.

It may be useful to analyse the historic prices for each region of NEM along with forecast prices depicted in the same graph as one realises the inappropriateness of this model in predicting electricity prices with a pure mean-reverting better with the aid of this type of
visual representation. The following graphs illustrate the actual and simulated prices for each region of the NEM.

Figure 3 Electricity Price Forecast with Mean-Reverting Process for NSW

![NSW daily log prices and forecast](image)

Figure 4 Electricity Price Forecast with Mean-Reverting Process for VIC

![VIC daily log prices and forecast](image)
Figure 2 Electricity Price Forecast with Mean-Reverting Process for SA

Figure 6 Electricity Price Forecast with Mean-Reverting Process for QLD
In order to assess the forecast accuracy of the model, the average of 10,000 simulated paths are taken as the point in time forecast value. As it is apparent in the visual illustration of forecast evaluation (forecast values versus out of sample data) depicted in the following chart, the mean-reverting model, first of all, fails to simulate the observed time series characteristics of the electricity prices in generating spikes. Secondly, the model tends to overestimate the general price trend as the forecast values are most of the time above the actual prices.
Figure 8 Forecast Versus Actual Prices

Actuals versus simulated prices for NSW region

- Actuals
- Forecast